Contents lists available at ScienceDirect

International Review of Financial Analysis

journal homepage: www.elsevier.com/locate/irfa

GARCH-M model with an asymmetric risk premium: Distinguishing between 'good' and 'bad' volatility periods

Juri Trifonov *, Bogdan Potanin

HSE University, Pokrovsky Blvd 11, 109028, Moscow, Russia

ARTICLE INFO

JEL classification:
C18
C22
C58

Keywords:
GARCH
Leverage effect
Risk premium
Conditional volatility

ABSTRACT

We proposed a new method (GARCH-M-GJR-LEV) that captures the asymmetry in the variance and return equations. The development of the model is encouraged by the stylized fact that investors demand a higher risk premium during "bad" volatility periods rather than "good" ones. To study the properties of the obtained estimators, we conducted simulated data analysis, considering a data-generating process characterized by asymmetric responses of risk premium to volatility changes. As a result, we have found statistical evidence in favor of a significant advantage of the proposed method compared to existing alternatives. Further, the proposed model was applied to study the S&P 500 market index. We have found evidence of an asymmetric relationship between the risk premium and volatility changes during most periods under consideration. Due to this, the GARCH-M-GJR-LEV model usually outperformed the alternative GARCH family models according to the information criteria.

1. Introduction

Volatility is considered one of the essential factors in financial markets. The indicator reflects assets' risk level and is usually measured by the standard deviation of returns. Thus, volatility modeling is indispensable in risk management policy, option pricing models, and other fields of finance (Miralles-Marcelo et al., 2013). The GARCH family models, first introduced by Bollerslev (1986) and Engle (1982), are supposed to be the most common method of modeling conditional volatility of financial assets.

The introduction of GARCH encouraged the development of a new literature stratum dedicated to creating various GARCH-type models. These modifications of classical GARCH were stimulated by the need to adapt the model to the financial theory and stylized facts in financial markets.

According to the main portfolio theories of Markowitz (1952) and Sharpe (1964), asset returns and volatility are the key factors that describe pricing. Specifically, returns and volatility have a positive correlation. Thus, investors demand higher returns when the asset is more volatile. This idea has become the foundation for introducing the central financial economics concept. The concept states that every risky asset incorporates a risk premium component in its return. Because classical GARCH models do not allow to account for a risk premium existence in the returns equation, this led to the development of a new GARCH-type model. The GARCH-in-Mean (GARCH-M) model has been introduced by Engle et al. (1987). Its key feature was incorporating the conditional variance into the mean equation, introducing the contribution of the risk premium to the dynamics of returns. This model has become widely used by researchers. Nevertheless, Bollerslev (2022) indicates that inconsistent results concerning the sign of the risk premium and its robustness have been found in several papers: Bollerslev et al. (2006), Hong and Linton (2020) and Rossi and Timmermann (2015). According to Bollerslev (2022), recent studies illustrate contradictory outcomes since the contribution sign of the risk premium differs among the periods and assets. Moreover, he suggests that the asymmetric relation between volatility and return may cause such inconsistency. The classical GARCH-M model does not allow to capture such asymmetry. Thus, the estimator of the risk premium contribution may be inconsistent.

Although the concept of asymmetric relation is novel in the framework of the GARCH-M model, it has been extensively studied in the context of classical GARCH models. It refers to the asymmetric responses of conditional variance to shocks in returns. There is a widely known stylized fact in financial markets that volatility reacts more sharply to negative shocks in returns than positive ones (Black, 1976; Zhang, 2006). Because classical GARCH is symmetric, it does not allow to account for such effects (Nelson, 1991). This problem has led to the development of leveraged GARCH models that account for the asymmetry effect of random shocks in the variance equation. Most popular models of this type, including EGARCH (Nelson,

* Corresponding author.
E-mail addresses: ju.trifonov@gmail.com (J. Trifonov), bogdanpotanin@gmail.com (B. Potanin).

https://doi.org/10.1016/j.irfa.2023.102941
Received 9 November 2022; Received in revised form 10 February 2023; Accepted 14 September 2023

1991) and GJR-GARCH (Glosten et al., 1993), have been focusing on the functional form of the asymmetry effect in the variance equation. Modern studies mainly concentrate on developing more flexible asymmetric specifications of variance equation (Hansen et al., 2012) or on the multivariate generalization of classical asymmetric GARCH models (McAleer et al., 2009). However, the concept of capturing asymmetric responses of returns to conditional volatility, i.e., the asymmetry in the mean equation, is poorly studied in the literature. Our paper fills this gap by introducing the leverage effect in the mean part of the GARCH-M model.

In this study, we propose a new GARCH-M-GJR-LEV model that accounts for the asymmetry effect not only in the variance equation but also in the equation of returns. The model is constructed in the framework of GJR-GARCH and is based on the idea that negative shocks in returns lead to the higher contribution of conditional volatilities to asset returns. In other words, the model includes an additional parameter in the mean equation that reflects the asymmetric responses of returns to volatility changes.

According to the simulated data analysis results, we have found statistical evidence in favor of the significant advantage of the proposed method in comparison with other GARCH-type models. Moreover, using the log returns of the S&P 500 index, we have found evidence in favor of an asymmetric risk premium existence over different time intervals.

The paper proceeds as follows: In the next section, we briefly introduce and discuss GARCH-M and GJR-GARCH specifications. Next, we introduce our model, explain the intuition and derive the unconditional variance of returns. In Section 4, we provide the results of the simulated data analysis. In Section 5, we apply our model to study the log returns of the S&P 500 index and discuss the results. Finally, we summarize our contribution and findings.

2. The GARCH-M model

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let us define the σ-algebra in $t-1$ period as $\mathcal{F}_{t-1}$, which represents the set of all available information up to period $t-1$, $t \in \mathbb{N}$. Let us denote y as a $T \times 1$ vector of log returns and σ as a $T \times 1$ vector of conditional volatilities, where $T \in \mathbb{N}$. Then without loss of generality, we can specify the GARCH-M(1,1) (Engle et al., 1987) process with the following system of equations:

$$
\begin{gathered}
y_t = \mu + \lambda\sigma_t + \varepsilon_t, \\
\varepsilon_t|\mathcal{F}_{t-1} \sim \mathcal{N}(0, \sigma_t^2), \\
Var(y_t|\mathcal{F}_{t-1}) = \sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2, \\
\xi_t \sim \mathcal{N}(0,1) \ \ i.i.d., \\
\varepsilon_t = \sigma_t\xi_t,
\end{gathered} \tag{2.1}
$$

where parameters ω, α, and β determine the dynamics of conditional variance and $t \in \{1, \dots, T\}$. The distinctive feature of the model is the presence of σ_t in the mean equation. Therefore, volatility is allowed to influence returns directly, and, thus, $\lambda\sigma_t$ is usually interpreted as a risk premium. Based on the portfolio theories of Markowitz (1952) and Sharpe (1964), λ is expected to be positive because the increase in volatility leads to a higher return (Engle et al., 1987).

Although the model allows for capturing a risk premium effect, the specification does not allow to account for the asymmetric responses. In other words, returns react identically to changes in volatility without considering the sign of the shocks that caused it. Therefore, volatility during both "bear" and "bull" markets has the same effect on returns.

3. The GJR-GARCH model

The GJR-GARCH (Glosten et al., 1993) model has been developed to capture the leverage effect in financial markets. We use the GJR-GARCH framework as the basis for our model since its theoretical properties are well studied even in the multivariate case (McAleer et al., 2009). The other possible alternative is EGARCH proposed by Nelson (1991). According to the empirical study of Awartani and Corradi (2005), EGARCH has demonstrated the highest accuracy on S&P 500 index data compared to various asymmetric GARCH-family models, including GJR-GARCH, TGARCH, and AGARCH. However, it is complicated to derive the theoretical properties of the EGARCH model since it is based on a complex random coefficient nonlinear moving average process whose stationarity conditions are still unknown (McAleer & Hafner, 2014). Moreover, according to Awartani and Corradi (2005), GJR-GARCH performed as the second best (after EGARCH) when the author studied combinations of asymmetric GARCH models with a classical GARCH-M. Therefore we prefer GJR-GARCH over other asymmetric GARCH specifications as a starting point for our model because it is relatively easy to derive theoretical properties of its modifications (i.e., of GARCH-M-GJR-LEV), and it has demonstrated reasonably good performance on S&P 500 index that we use in the empirical part of our study.

This process is identical to the classical GARCH specification but allows for asymmetric variance responses to shocks in returns. Following the notations from the previous model, the GJR-GARCH(1,1) process is specified as follows:

$$
\begin{gathered}
y_t = \mu + \varepsilon_t, \\
\varepsilon_t|\mathcal{F}_{t-1} \sim \mathcal{N}(0, \sigma_t^2), \\
Var(y_t|\mathcal{F}_{t-1}) = \sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma I_{t-1}\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2, \\
I_t = \begin{cases} 0, \text{ if } \varepsilon_t \geq 0 \\ 1, \text{ if } \varepsilon_t < 0, \end{cases} \\
\xi_t \sim \mathcal{N}(0,1) \ \ i.i.d., \\
\varepsilon_t = \sigma_t\xi_t.
\end{gathered} \tag{3.1}
$$

The crucial feature of this model is that volatility may react differently to positive and negative shocks in returns. It is implemented by including an indicator variable I_t that reflects the sign of the shock in period t. Thus, it may be easily seen that if the shock is positive, the contribution to volatility will be made only by the α parameter. In contrast, negative shocks reflect a sharper contribution to conditional volatility equal to $\alpha + \gamma$.

It is straightforward to show that the unconditional mean and variance of the process are specified as follows:

$$
\begin{gathered}
\mathbb{E}[y_t] = \mu, \\
Var\left(y_t\right) = \sigma^2 = \frac{1}{1 - \alpha - \gamma/2 - \beta}.
\end{gathered} \tag{3.2}
$$

By definition, the unconditional variance is always nonnegative. Therefore, we can easily impose the parameter constraints that ensure the existence of a stationary solution:

$$
\begin{gathered}
\omega > 0, \\
\alpha + \gamma/2 + \beta < 1.
\end{gathered} \tag{3.3}
$$

The model has been developed to account for the stylized fact about the asymmetric relationship between volatility and shocks in asset returns. According to Black (1976) and Zhang (2006), statistical evidence has been found that volatility reacts more sharply to negative shocks in returns than positive ones. Therefore, the sign of the γ parameter, reflecting the leverage effect, is expected to be positive. The positive estimate will reflect more considerable volatility growth when the observed shocks are negative. Several approaches exist in the literature to explain the causes of the asymmetry effect. According to Black (1976) and Christie (1982), negative shocks lead to an increase in the financial leverage of issuing companies. Consequently, higher leverage enlarges the stocks' level of risk and leads to an increase in their volatility. Moreover, according to the prospect theory of Kahneman and Tversky (1979), the asymmetry effect can arise in the framework of investors' cognitive features. People tend to perceive losses more critically, and, therefore, negative shocks in returns may result in massive asset sales by investors, thereby provoking an increase in volatility.

4. The GARCH-M-GJR-LEV model

The classical GARCH-M model allows capturing risk premium in assets' returns. Albeit, the model does not allow for asymmetric responses of returns to risk premium fluctuations. Therefore, the process cannot differentiate between signs of shocks that cause the volatility, i.e., it equally perceives negative and positive shocks. Thus, the GARCH-M assumes that investors demand equal risk premiums when the market experiences “good” and “bad” volatility periods. In other words, the model accounts only for the absolute value of shocks determining the risk premium. Nevertheless, according to Bollerslev (2022), statistical evidence has been found that GARCH-M may demonstrate insignificant risk premiums in asset returns. Moreover, some studies have shown that the model may provide negative risk premium estimates. Such results are inconsistent with the main portfolio theories of Markowitz (1952) and Sharpe (1964) since they state an inverse risk-return relationship, implying a negative correlation between volatility and returns. Following Bollerslev (2022), the problem may be explained by the necessity of differentiation between “good” and “bad” volatility measures. Rational investors tend to demand a higher risk premium during the “bad” volatility period (downside risk) than in the case of the “good” period (upside potential).

This article proposes a new class of GARCH models that accounts for an asymmetric relationship between returns and volatility. The distinctive feature of the proposed method is that it allows for differentiation between “good” and “bad” volatility periods in the returns equation. We construct the model in the framework of the GJR-GARCH process. Thereby, it captures leverage effects both in the return and variance equations.

Following the notations from Sections 2 and 3, we specify the proposed model as follows:

$$\begin{gathered} y_t = \mu + \lambda_1 \sigma_{t-1}^2 + \lambda_2 I_{t-1} \sigma_{t-1}^2 + \varepsilon_t, \\ \varepsilon_t | \mathcal{F}_{t-1} \sim \mathcal{N}(0, \sigma_t^2), \\ Var(y_t|\mathcal{F}_{t-1}) = \sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma I_{t-1}\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2, \\ I_t = \begin{cases} 0, \text{ if } \varepsilon_t \geq 0 \\ 1, \text{ if } \varepsilon_t < 0, \end{cases} \\ \varepsilon_t = \sigma_t \xi_t, \\ \xi_t \sim \mathcal{N}(0,1) \ i.i.d., \end{gathered} \tag{4.1}$$

where parameters λ_1 and λ_2 define the influence of conditional variance on returns. The presence of $\lambda_2 I_{t-1}\sigma_{t-1}^2$ allows differentiating between “good” and “bad” volatility periods while defining the risk premium. Therefore, volatility caused by negative shocks in returns can lead to a higher risk premium. As a consequence, we expect the estimate of λ_2 to be higher than λ_1. When $\lambda_2 > 0$, it reflects the idea that investors demand a higher risk premium when the volatility is caused by a “bear” market.

The model combines the GARCH-M process with the GJR-GARCH specification. Although, we use a slightly modified version of the GARCH-M model, imposing two specific differences. First, we specify the risk premiums through the conditional variances as $\lambda_i\sigma_{t-1}^2$, $i = \{1,2\}$, unlike the classical GARCH-M process that defines it using the conditional volatility, i.e., $\lambda\sigma_t$. We specify it this way since the unconditional variance of the proposed method may be explicitly derived only by including a conditional variance instead of conditional volatility. Second, while the classical GARCH-M model uses the current volatility to specify a risk premium, we use the variance of the previous period, i.e., σ_{t-1}^2. It is necessary since the last period of conditional variance is required to estimate the model. We may clearly see that ε_t defines the binary variable I_t in the same period t. In other words, if we specify the model with σ_t^2 and $I_t\sigma_t^2$ (instead of σ_{t-1}^2 and $I_{t-1}\sigma_{t-1}^2$), it will be impossible to define ε_t in the same period and, therefore, estimate the model via the maximum likelihood method.

Also, when the parameters λ_2 and γ are equal to zero, then the specification converges to the process that is very similar to the classical GARCH-M:

$$\begin{gathered} y_t = \mu + \lambda_1 \sigma_{t-1}^2 + \varepsilon_t, \\ \varepsilon_t | \mathcal{F}_{t-1} \sim \mathcal{N}(0, \sigma_t^2), \\ Var(y_t|\mathcal{F}_{t-1}) = \sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2, \\ \varepsilon_t = \sigma_t \xi_t, \\ \xi_t \sim \mathcal{N}(0,1) \ i.i.d. \end{gathered} \tag{4.2}$$

Further, to impose the necessary stationarity conditions, we need to specify the unconditional variance of returns with Theorem 1. The complete proof of the Theorem is given in Appendix.

Theorem 1. *Suppose there exists a stationary solution to the proposed system of equations. Then, the expression for the unconditional variance of returns in the GARCH-M-GJR-LEV model has the following form:*

$$\begin{aligned} Var(y_t) = \sigma^2 = (\lambda_1^2 + \lambda_1\lambda_2) \times \left(\mathbb{E}\left[\sigma_t^4\right] - \mathbb{E}\left[\sigma_t^2\right]^2\right) \\ + \frac{1}{2}\lambda_2^2 \times \left(\mathbb{E}\left[\sigma_t^4\right] - \frac{1}{2}\mathbb{E}\left[\sigma_t^2\right]^2\right) + \mathbb{E}\left[\sigma_t^2\right], \end{aligned} \tag{4.3}$$

where $\mathbb{E}\left[\sigma_t^4\right]$ *is an expectation of* σ_t^4 *and is defined as follows:*

$$\mathbb{E}\left[\sigma_t^4\right] = \frac{\omega^2 + \omega\mathbb{E}\left[\sigma_t^2\right] \times (2\alpha + 2\beta + \gamma)}{1 - 3\alpha^2 - \beta^2 - \frac{3}{2}\gamma^2 - 2\alpha\beta - 3\alpha\gamma - \beta\gamma}. \tag{4.4}$$

Also, $\mathbb{E}\left[\sigma_t^2\right]$ *denotes the unconditional variance of* ε_t*:*

$$\sigma_\varepsilon^2 = Var(\varepsilon_t) = \frac{\omega}{1 - \alpha - \gamma/2 - \beta}. \tag{4.5}$$

By the definition of variance, we need to ensure that the value of Eq. (4.3) is positive. Thus, the stationary solution exists if and only if:

$$\sigma^2 = Var(y_t) > 0. \tag{4.6}$$

Usually, GARCH models are estimated via the maximum likelihood estimator. The log-likelihood function of the GARCH-M-GJR-LEV model may be specified as follows, assuming a normal distribution of random shocks:

$$\ln L(\theta, \varepsilon) = -\frac{T}{2}\ln 2\pi - \frac{1}{2}\sum_{t=1}^{T}\ln\sigma_t^2 - \frac{1}{2}\sum_{t=1}^{T}\frac{\varepsilon_t^2}{\sigma_t^2}, \tag{4.7}$$

where $\theta = (\mu, \omega, \alpha, \beta, \lambda_1, \gamma, \lambda_2)'$ is a vector of estimated parameters, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_T)'$. The values of conditional volatilities σ_t and random shocks ε_t may be calculated recursively, following the GARCH-M-GJR-LEV process:

$$\begin{aligned} \varepsilon_t|\mathcal{F}_{t-1} = y_t - \mu - \lambda_1\sigma_{t-1}^2 - \lambda_2 I_{t-1}\sigma_{t-1}^2, \\ \sigma_t^2|\mathcal{F}_{t-1} = \omega + \alpha\varepsilon_{t-1}^2 + \gamma I_{t-1}\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2. \end{aligned} \tag{4.8}$$

Maximization of the log-likelihood function provides estimators for the unknown parameters $\mu, \omega, \alpha, \beta, \lambda_1, \gamma$, and λ_2.

5. Simulated data analysis

We conducted simulated data analysis to study the properties of the obtained estimators and compare them with alternatives. The main point under consideration was to determine whether it is crucial to account for asymmetric responses in the mean equation or whether other models may provide accurate estimates of the parameters, volatilities, and returns even under the GARCH-M-GJR-LEV data generating process. The parameters of the simulated data analysis are described in Table 1.

We considered two sets of parameter values: Set I and Set II. In Set I, we set the true values of parameters similar to those frequently observed in applications of GARCH models to stock returns. Parameters responsible for leverage effects, both in the mean and variance

Table 1
Parameters of simulations.

Parameter	Set I	Set II
μ	0.01	0.05
ω	0.1	0.05
α	0.1	0.05
β	0.7	0.8
λ_1	0.2	−0.05
γ	0.15	0.2
λ_2	0.5	0.2
Sample size	1000	
Number of simulations	100	

equations, were set to the values that significantly change the data-generating process. In Set II, parameters are assigned to the values that replicate our results of the GARCH-M-GJR-LEV model application to the S&P 500 index stock returns (see Section 6). Next, we decided to use a sample size of 1000 observations. We chose such a sample size to make the experiment similar to the real data application problems that usually provide a long enough time series of returns. At the same time, larger samples are generally unavailable due to structural breaks. Consequently, the experiment is designed to replicate practical applications as closely as possible. Besides that, such a sample size allows for studying the large sample properties of estimators. Finally, the total number of simulations equals 100 since the share of simulations where the proposed method has demonstrated a significant advantage over the analogs is significantly high (see Tables 4–5). At the same time, a higher number of simulations would require extensive computational and time resources.

The simulated data analysis proceeds as follows. First, we generate a pseudo-random sample following the GARCH-M-GJR-LEV data-generating process. Second, we use the generated data to estimate three models: GARCH-M, GARCH-M-GJR, and GARCH-M-GJR-LEV.[1] By obtaining estimates of each model, we compare their accuracy and properties by using information criteria and accuracy metrics.

It is important to note that the likelihood function in the GARCH-M-GJR-LEV model may not be necessary concave. Therefore, to reduce the risks that the optimization procedure may converge to a local maximum, we use 100 iterations of the hybrid genetic algorithm of global optimization with BFGS local optimizer that takes place each iteration with a probability of 0.1.[2] To get an initial point for the optimization routine for the GARCH-M model, we set $\lambda_1 = 0$ and take classical GARCH estimates for other parameters. Similarly, we take GARCH-M estimates as initial points for GARCH-M-GJR, setting $\gamma = 0$. Finally, we take GARCH-M-GJR estimates as the initial point for the GARCH-M-GJR-LEV model, setting $\lambda_2 = 0$.

We calculated *RMSE*[3] values based on the coefficient estimates to compare the estimation accuracy among all four models. We use the following formula to calculate *RMSE* metrics for every coefficient estimate:

$$RMSE(\hat{\theta}) = \sqrt{\frac{1}{100}\sum_{m=1}^{100}\left(\theta_m - \hat{\theta}_m\right)^2}, \tag{5.1}$$

Table 2
Accuracy metrics of coefficient estimates (Set I).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$RMSE(\hat{\mu})$	7.091	6.056	4.616
$RMSE(\hat{\omega})$	3.390	3.111	2.609
$RMSE(\hat{\alpha})$	10.106	4.013	4.144
$RMSE(\hat{\beta})$	6.147	5.885	5.403
$RMSE(\hat{\lambda}_1)$	21.438	7.657	6.639
$RMSE(\hat{\gamma})$	–	22.917	6.513
$RMSE(\hat{\lambda}_2)$	–	–	8.934

Table 3
Accuracy metrics of coefficient estimates (Set II).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$RMSE(\hat{\mu})$	4.196	3.283	3.383
$RMSE(\hat{\omega})$	3.150	1.694	1.700
$RMSE(\hat{\alpha})$	12.824	3.092	3.168
$RMSE(\hat{\beta})$	7.106	4.095	3.965
$RMSE(\hat{\lambda}_1)$	14.072	26.764	25.596
$RMSE(\hat{\gamma})$	–	15.310	23.970
$RMSE(\hat{\lambda}_2)$	–	–	6.997

where $\theta = \{\mu, \omega, \alpha, \beta, \lambda_1, \gamma, \lambda_2\}$ denotes a set of estimated parameters, and m is an index of each simulation.

The values of *RMSE* metrics for every estimate in each model are presented in Tables 2 and 3 (for Set I and Set II correspondingly).

Based on the obtained results for Set I, we can see that the GARCH-M-GJR-LEV model provides significantly lower values of the *RMSE* metric for all coefficient estimates except $\hat{\alpha}$, which is slightly more accurate in the case of the GARCH-M-GJR model. Consequently, we have found statistical evidence in favor of the advantage of the proposed method in terms of the accuracy of estimates. Following the results in Table 2, if the true data generating process differs from the one assumed by the model, the obtained estimators may become inaccurate without accounting for the leverage effect. The results for Set II (Table 3) reveal that the proposed method allows estimating the λ_1 parameter more accurately by capturing the leverage effect in the risk premium. Furthermore, according to the *RMSE* value for $\hat{\lambda}_2$, the proposed model tends to estimate λ_2 accurately (even more precisely than λ_1). This evidence justifies the development of the GARCH-M-GJR-LEV model because if the observed data demonstrate an asymmetric relationship between risk premium and volatility, then existing methods are likely to provide inefficient and inaccurate estimates.

Although parameter estimates are vital, the accuracy of predicted volatilities and returns plays a more important role in practical applications. Therefore, we calculated *RMSE* values for predicted conditional volatilities and returns.[4] We evaluated *RMSE* metrics for in-sample predictions for each simulation in every model. To aggregate the results, we calculated the mean values of *RMSE* among all simulations using the following formulas:

$$\begin{aligned} RMSE(\hat{\sigma}) &= \frac{1}{100}\sum_{m=1}^{100}\sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\sigma_t - \hat{\sigma}_t\right)^2}, \\ RMSE(\hat{y}) &= \frac{1}{100}\sum_{m=1}^{100}\sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(y_t - \hat{y}_t\right)^2}. \end{aligned} \tag{5.2}$$

The mean values of *RMSE* for conditional volatilities and returns are presented in Tables 4–5 for Set I and Set II correspondingly. Besides that, we provide shares of simulations where each specific model has demonstrated the lowest values of the *RMSE* criterion for conditional volatilities and returns *(Victories %)*. Finally, the average values of Akaike *(AIC)* and Bayesian *(BIC)* information criteria are calculated for every model.

[1] We use the modified version of the GARCH-M model, defining the risk premiums as $\lambda\sigma_{t-1}^2$ (see Section 4). Such modification is applied for the straightforward comparison with the GARCH-M-GJR-LEV model since, in this case, GARCH-M is nested inside the GARCH-M-GJR-LEV process.

[2] During the preliminary analysis, we found that after 100 iterations genetic algorithm rarely provides notable improvements to the log-likelihood value for the models under consideration. Although, sometimes, these 100 iterations provide significant improvements in terms of log-likelihood value and estimation accuracy.

[3] For convenience, we indicate *RMSE* values multiplied by 100. Please, see Appendix B for alternative metrics.

[4] We also considered the *MAE* and *MSE* metrics. However, we moved them into the Appendix since they indicated similar results.

Table 4
Accuracy metrics and information criteria (Set I).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$RMSE(\hat{\sigma})$	12.345	10.295	6.142
$Victories_{\sigma}\%$	0%	4%	96%
$RMSE(\hat{y})$	94.581	94.210	90.568
$Victories_{y}\%$	0%	0%	100%
AIC	2573.072	2559.644	2500.384
BIC	2597.610	2589.091	2534.738

Table 5
Accuracy metrics and information criteria (Set II).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$RMSE(\hat{\sigma})$	13.290	6.938	5.804
$Victories_{\sigma}\%$	0%	19%	81%
$RMSE(\hat{y})$	100.393	100.285	98.766
$Victories_{y}\%$	0%	4%	96%
AIC	2619.222	2598.067	2585.704
BIC	2643.761	2627.514	2620.058

The results clearly show that the proposed method provides significantly lower values of the *RMSE* criterion. For Set I, The GARCH-M-GJR-LEV model demonstrates about two times the lower value of *RMSE* for conditional volatilities than alternative models. Moreover, the share of lower *RMSE* values is much higher for the proposed method: the GARCH-M-GJR-LEV has provided more accurate conditional volatility forecasts in 92% cases and more accurate forecasts of returns in 98% cases correspondingly. The results for Set II similarly reveal that the developed model estimate volatilities and returns more precisely than its counterparts. Thus, the results of simulated data analysis suggest a significant advantage of the proposed method and justify its development and application. In other words, if the data generating process involves the asymmetric relationship between risk premium and volatility, the existing methods may demonstrate inefficient estimators and significantly lower accuracy of forecasts for volatilities and returns.

Finally, we have tested the robustness of the maximum likelihood estimator of the GARCH-M-GJR-LEV model to distributional assumption violations.[5] It is well known that a classical GARCH process estimator (based on the normality assumption) may preserve consistency even if random shocks' distribution deviates from normality (Berkes & Horváth, 2004). Unfortunately, establishing a similar formal result for the GARCH-M-GJR-LEV model is a technically complicated task beyond this paper's scope. So, instead, we test the finite sample robustness of the GARCH-M-GJR-LEV estimator by replicating the analysis for Set 1, alternately assuming that ξ_t follows the Student's t and noncentral Student's t-distributions (standardized to zero mean and unit variance). To ensure that tails are rather heavy, we use 5 degrees of freedom for both distributions. Noncentrality parameter for the noncentral Student's t-distribution also has been set to 5, which makes this distribution positively skewed. The results are presented in the Appendix (Tables C.18–C.21) and suggest that distributional assumption violation provides just a slight increase in *RMSE* values. It indicates in favor of the finite sample robustness of the GARCH-M-GJR-LEV estimator.

6. Application to the S&P 500 index returns

After studying the properties of the proposed method, we applied it to the actual data, represented by the log returns of the S&P 500 market index. The sample consists of 4531 observations and covers the period from 01.01.2004 to 31.12.2021. Fig. 1 represents the dynamics of the S&P 500 log returns. The data was retrieved from the Bloomberg Terminal.

[5] In the case of distributional assumption violation, it becomes a quasi-maximum likelihood estimator (QMLE).

Table 6
S&P 500 estimation results for the period 2004–2021.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
μ	0.0499***	0.0225	0.0347**
	(0.0141)	(0.0138)	(0.0140)
ω	0.0286***	0.0287***	0.0291***
	(0.0024)	(0.0020)	(0.0012)
α	0.1347***	0.0129***	0.0162***
	(0.0084)	(0.0042)	(0.0030)
β	0.8389***	0.8589***	0.8553***
	(0.0092)	(0.0080)	(0.0005)
λ_1	0.0265*	0.0135	−0.0237***
	(0.0137)	(0.0128)	(0.0091)
γ	–	0.1903***	0.1871***
		(0.0120)	(0.0091)
λ_2	–	–	0.0760***
			(0.0003)
AIC	**11 791.596**	**11 643.245**	**11 629.996**

Note: *** – $p < 0.01$, ** – $p < 0.05$, * – $p < 0.1$; st.errors in parentheses.

To ensure the robustness of the results, we estimated models for the entire sample and specific periods. It is essential to consider that long financial time series may be characterized by structural breaks. Therefore, we decided to estimate the model on a set of samples that include observations of 3 years. The main goal of the analysis is to examine the S&P 500 index for the asymmetric responses of risk premium to conditional volatility and compare the results of the GARCH-M-GJR-LEV model with alternatives by the Akaike information criterion. The estimation results for the whole sample and specific periods are presented in Tables 6–13. The covariance matrix was estimated via the gradient outer product (GOP).

6.1. Whole sample analysis

In this subsection, we provide estimation results for the entire sample. To interpret the results meaningfully, it is necessary to evaluate all three models: GARCH-M, GARCH-M-GJR, and GARCH-M-GJR-LEV. Let us recall that the GARCH-M model accounts for the risk premium effect, and GARCH-M-GJR captures the risk premium effect along with the leverage effect in the volatility equation. Finally, the proposed GARCH-M-GJR-LEV model allows accounting for the asymmetry effect both in volatility and risk premium responses to shocks in returns.

Based on the results presented in Table 6, all parameters in the GARCH-M-GJR-LEV model are statistically significant at any reasonable level. The parameters of most interest are λ_1 and λ_2. The estimate of λ_1 has a negative sign which is not intuitive since returns appear to react negatively to volatility increases. Although this coincides with the results of previous research by Bollerslev (2022), such evidence may be caused by structural breaks. Therefore, we analyze this effect more carefully by estimating smaller samples in the next subsections. The estimate of λ_2 is positive and demonstrates that risk premium increases more sharply when volatility rises due to the negative shocks in returns.

The asymmetric relationship between the risk premium and previous shocks in returns is visualized in Fig. 2. Since the estimate of λ_1 is negative and the estimate of λ_2 is positive, risk premium only rises when the observed shocks in returns are negative. This result clearly illustrates the above-mentioned contradictory findings and demonstrates that during "good" volatility periods, investors tend to demand a discount instead of a premium.

Comparing the results with the other two models, we see that estimate of λ in the GARCH-M model is positive and statistically significant. Although, the absolute value of the effect is much smaller than in the GARCH-M-GJR model. Consequently, by estimating GARCH-M instead of the GARCH-M-GJR-LEV process, a researcher may underestimate the risk premium. Moreover, applying the GARCH-M-GJR model, we identify a significant leverage effect in the volatility equation (γ), while the risk premium effect becomes insignificant and small. Therefore,

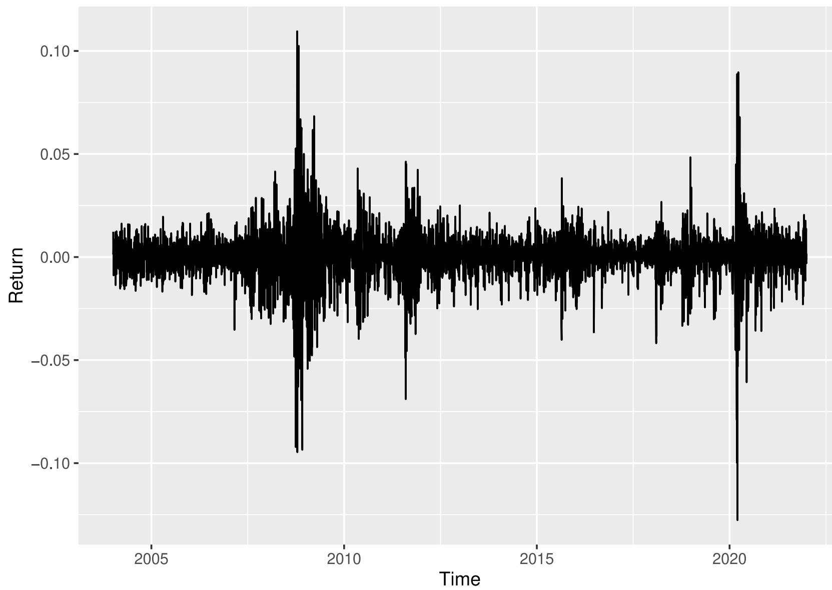

Fig. 1. Dynamics of the S&P 500 market index log returns.

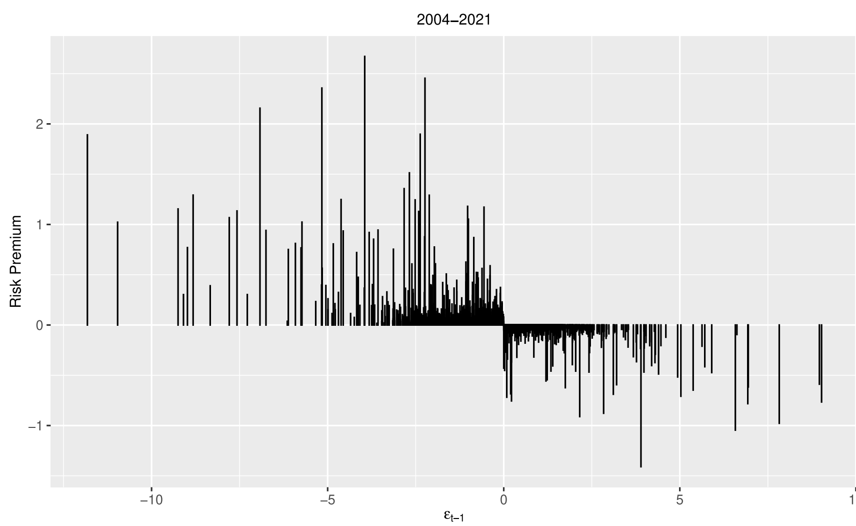

Fig. 2. Risk premium responses to shocks in returns.

accounting for the leverage effect in the variance equation, one can misidentify the absence of the risk premium in returns since the leverage coefficient in the volatility equation takes on all the influence. Consequently, to correctly identify the risk premium and leverage effects, the proposed GARCH-M-GJR-LEV model should be used because of the presence of an asymmetric risk premium effect in the observed data. Finally, the Akaike *(AIC)* criterion identifies the method as the best of all three options.

6.2. Analysis of 3-year samples: 2019–2021, 2018–2020, 2017–2019

This subsection provides results for the last three sequential 3-year samples. The estimation results are presented in Tables 7–9 correspondingly for each period. We combine these three periods into one subsection since they demonstrate similar results.

All three periods are characterized by statistically insignificant and small estimates of λ_1 and λ_2 parameters in the GARCH-M-GJR-LEV model. Consequently, we have found no statistical evidence of risk premium effects in the S&P 500 returns on the analyzed periods. We should note that this finding coincides with the results of the GARCH-M and

Table 7
S&P 500 estimation results for the period 2019–2021.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
μ	0.0994***	0.0762**	0.0801**
	(0.032)	(0.0319)	(0.0327)
ω	0.0656***	0.0718***	0.0711***
	(0.0108)	(0.0115)	(0.0115)
α	0.3034***	0.1623***	0.1616***
	(0.0321)	(0.0231)	(0.023)
β	0.6777***	0.6569***	0.6574***
	(0.0296)	(0.0297)	(0.0298)
λ_1	0.0325	0.0149	−0.0114
	(0.0219)	(0.019)	(0.0271)
γ	–	0.3176***	0.3205***
		(0.0682)	(0.0682)
λ_2	–	–	0.0644
			(0.0405)
AIC	**2058.326**	**2046.602**	**2046.535**

Note: *** – $p < 0.01$, ** – $p < 0.05$, * – $p < 0.1$; st.errors in parentheses.

Table 8
S&P 500 estimation results for the period 2018–2020.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
μ	0.1017***	0.0780**	0.0829**
	(0.0314)	(0.0328)	(0.0328)
ω	0.0550***	0.0533***	0.0522***
	(0.0092)	(0.0094)	(0.0093)
α	0.2467***	0.1390***	0.1399***
	(0.0289)	(0.0208)	(0.0210)
β	0.7227***	0.7228***	0.7242***
	(0.0274)	(0.0261)	(0.0260)
λ_1	0.0185	0.0060	−0.0156
	(0.0242)	(0.0224)	(0.0097)
γ	–	0.2116***	0.2077***
		(0.0437)	(0.0426)
λ_2	–	–	0.0489
			(0.0317)
AIC	**2145.537**	**2132.454**	**2133.173**

Note: *** – $p < 0.01$, ** – $p < 0.05$, * – $p < 0.1$; st.errors in parentheses.

Table 9
S&P 500 estimation results for the period 2017–2019.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
μ	0.0911***	0.0616**	0.0566*
	(0.0300)	(0.0291)	(0.0301)
ω	0.0306***	0.0287***	0.0298***
	(0.0051)	(0.0045)	(0.0049)
α	0.2022***	0.0106	0.0134
	(0.0258)	(0.0204)	(0.0218)
β	0.7561***	0.7910***	0.7808***
	(0.0303)	(0.0277)	(0.0291)
λ_1	0.0390	0.0064	0.0243
	(0.0579)	(0.0594)	(0.0753)
γ	–	0.2856***	0.2928***
		(0.0357)	(0.0387)
λ_2	–	–	−0.0266
			(0.0789)
AIC	**1538.754**	**1500.615**	**1502.444**

Note: *** – $p < 0.01$, ** – $p < 0.05$, * – $p < 0.1$; st.errors in parentheses.

GARCH-M-GJR models because they do not capture a significant risk premium effect either. The result is also reflected in the slight difference in the *AIC* criterion between GARCH-M-GJR and GARCH-M-GJR-LEV models. Nevertheless, it is crucial to account for the leverage effect in the variance equation since both GARCH-M-GJR and GARCH-M-GJR-LEV models provide significant and positive estimates of γ, which evidences that volatility appears to react more sharply on negative shocks in returns than on positive ones. This conclusion is supported by the significant difference in the *AIC* criterion between the GARCH-M model and the two other methods. Overall, the analyzed three periods are characterized by the absence of risk premium. Still, they include the leverage effect in the volatility equation, and therefore, GARCH-M-GJR and the proposed GARCH-M-GJR-LEV model provide similar results.

6.3. Analysis of 3-year samples: 2016–2018, 2015–2017, 2014–2016, 2013–2015

In this subsection, we discuss results obtained from the analysis of four sequential 3-year samples, covering the period from 2013 to 2018. The estimation results of all periods are presented in Tables 10–13. As with the previous case, this section merges periods with similar patterns.

Based on the presented results, all estimated periods demonstrate statistically significant estimates of λ_2 parameter in the GARCH-M-GJR-LEV model. The result provides evidence that returns include the asymmetric relationship between the risk premium and shocks in returns, and the estimate is positive among all the estimated periods. Consequently, this clearly represents that risk premium increases more sharply when volatility rises due to negative shocks in returns

Table 10
S&P 500 estimation results for the period 2016–2018.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
μ	0.0598**	0.0301	0.0470
	(0.0297)	(0.0304)	(0.0301)
ω	0.0394***	0.0370***	0.0344***
	(0.0057)	(0.0054)	(0.0051)
α	0.2146***	0.0507***	0.0581***
	(0.0213)	(0.0143)	(0.0171)
β	0.7382***	0.7634***	0.7701***
	(0.0297)	(0.0284)	(0.0288)
λ_1	0.0424	0.0319	−0.0749
	(0.0573)	(0.0544)	(0.0525)
γ	–	0.2556***	0.2527***
		(0.0298)	(0.0398)
λ_2	–	–	0.1914***
			(0.0483)
AIC	**1575.468**	**1557.305**	**1553.541**

Note: *** – $p < 0.01$, ** – $p < 0.05$, * – $p < 0.1$; st.errors in parentheses.

rather than positive ones. This evidence supports the "bad" and "good" volatility differentiation hypothesis of Bollerslev (2022). That is, investors tend to demand a higher risk premium during the "bear" market, i.e., driven by "bad" volatility than in the case of "good" volatility periods.

According to Table 10, the parameter λ_1 in the GARCH-M model is statistically insignificant, while the GARCH-M-GJR-LEV model provides a significant λ_2 parameter. It is an important result since it demonstrates that the GARCH-M model could not capture a risk premium, while GARCH-M-GJR-LEV has identified it. Considering that the parameter λ_1 in the proposed method is also statistically insignificant, we can conclude that the risk premium in the S&P 500 returns on this period reacts only to "bad" volatility. In contrast, "bull" market volatility fluctuations do not increase the risk premium. In other words, investors demand risk premiums only during the drops in financial markets. In contrast, high market turbulence during periods of sharp growth does not stimulate an investor to require a higher risk premium. This pattern demonstrates the irrationality of investors: one appears to perceive negative news more dramatically in comparison with positive ones. Such evidence is consistent with the prospect theory of Black (1976), Kahneman and Tversky (1979), Nelson (1991) and Zhang (2006).

Besides that, it is essential to note that the data is also characterized by asymmetric volatility responses to shocks in returns since GARCH-M-GJR and GARCH-M-GJR-LEV models demonstrate significant and positive estimates of γ. We may also see it from the lower values of the *AIC* criterion in asymmetric models compared to the GARCH-M model.

Finally, the proposed GARCH-M-GJR-LEV model appears to be the best based according to the Akaike information criteria. In other words, the model allowed to capture both leverage effects and, consequently, ensure a higher estimation accuracy.[6] Furthermore, since the GARCH-M model did not capture a significant risk premium effect, one may misidentify the presence of risk premium in the returns without applying the GARCH-M-GJR-LEV model.

Further, Tables 11–13 represent similar results. The only difference in these periods is that the λ_1 parameter in the GARCH-M model is statistically significant in all three tables. This means that the GARCH-M model captured a risk premium effect in contrast to the 2016–2018 period. Besides that, it is crucial to analyze the estimation results provided by the GARCH-M-GJR model. It is seen that among all three tables, after accounting for the asymmetry in the variance equation (through the application of the GARCH-M-GJR model), the λ_1 parameter becomes statistically insignificant. That is, a researcher may not be able to capture the risk premium by estimating the GJR specification.

[6] Leverage effects in returns and volatility equations.

Table 11
S&P 500 estimation results for the period 2015–2017.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
μ	0.0004	0.0011	0.0065
	(0.0313)	(0.0312)	(0.0164)
ω	0.0464***	0.0492***	0.0497***
	(0.0068)	(0.0086)	(0.0085)
α	0.2401***	0.0543***	0.0589***
	(0.0218)	(0.0134)	(0.0125)
β	0.6996***	0.7107***	0.7003***
	(0.0329)	(0.0403)	(0.0398)
λ_1	0.1664**	0.0892	0.0134
	(0.0650)	(0.0594)	(0.0324)
γ	–	0.3065***	0.3098***
		(0.0352)	(0.0351)
λ_2	–	–	0.1792***
			(0.0486)
AIC	**1553.335**	**1531.684**	**1523.906**

Note: *** – $p < 0.01$, ** – $p < 0.05$, * – $p < 0.1$; st.errors in parentheses.

Table 12
S&P 500 estimation results for the period 2014–2016.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
μ	0.0004	0.0004	0.0004
	(0.0422)	(0.0397)	(0.0419)
ω	0.0691***	0.0703***	0.0679***
	(0.0123)	(0.0137)	(0.0139)
α	0.2499***	0.0233	0.0058
	(0.0262)	(0.0152)	(0.0174)
β	0.6874***	0.6996***	0.7172***
	(0.0399)	(0.0454)	(0.0461)
λ_1	0.1435**	0.0578	−0.0049
	(0.0656)	(0.0577)	(0.0635)
γ	–	0.3630***	0.3711***
		(0.0429)	(0.0402)
λ_2	–	–	0.1826**
			(0.0747)
AIC	**1753.369**	**1720.883**	**1715.978**

Note: *** – $p < 0.01$, ** – $p < 0.05$, * – $p < 0.1$; st.errors in parentheses.

Consequently, this may lead to a substantial misinterpretation of the results.

Further, as with the 2016–2017 results, the λ_1 parameter in the GARCH-M-GJR-LEV model is statistically insignificant (Tables 11–13). This evidences again that investors demand a risk premium only during "bad" volatility periods when variance rises due to the sharp falls in returns.

The leverage effect (γ) in the variance equation is significant among all periods. This result is consistent between the GARCH-M-GJR and GARCH-M-GJR-LEV models and across periods. Consequently, the S&P 500 index returns exhibit the asymmetry effect, which should be captured appropriately to provide a better estimation quality.

In Fig. 3, we present the dependence of the risk premium on previous values of shocks in returns for each period. All periods demonstrate an asymmetric relationship between the risk premium and volatility changes, based on the sign of shocks that cause volatility rises. The graph for the period 2016–2018 represents a similar pattern to the whole sample dependence (Fig. 2), while three other periods exhibit a different structure. The difference is explained only by the sign of the λ_1 estimate. Because it was negative and high for the period 2016–2018 (relative to other periods), thus the plot illustrates that investors demand a risk premium only during "bad" volatility periods, while "good" volatility may even produce a discount. The period 2015–2017 also illustrates that the risk premium is negative (risk discount) when shocks in returns are positive, albeit the absolute discount value is significantly lower compared to 2016–2018. It is due to the lower value of $\hat{\lambda}_1$. The remaining two periods demonstrate that risk premium rises during both "good" and "bad" volatility periods. At the same time, negative shocks in returns influence the premium more sharply, i.e., investors demand a higher risk premium during the "bear" market.

Table 13
S&P 500 estimation results for the period 2013–2015.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
μ	0.0042	0.0041	0.0174
	(0.0514)	(0.0387)	(0.0282)
ω	0.0750***	0.0683***	0.0631***
	(0.0194)	(0.0159)	(0.0144)
α	0.1973***	0.0002	0.0003
	(0.0374)	(0.0379)	(0.0069)
β	0.6854***	0.6954***	0.7039***
	(0.0524)	(0.0493)	(0.0461)
λ_1	0.1671*	0.0794	0.0123
	(0.0877)	(0.0625)	(0.0715)
γ	–	0.3996***	0.3915***
		(0.0588)	(0.0411)
λ_2	–	–	0.1477***
			(0.0476)
AIC	**1700.135**	**1651.677**	**1650.606**

Note: *** – $p < 0.01$, ** – $p < 0.05$, * – $p < 0.1$; st.errors in parentheses.

To summarize, statistical evidence has been found that the observed data exhibit the asymmetry effect in the variance equation and asymmetric responses of the risk premium to volatility changes. The proposed GARCH-M-GJR-LEV method allows capturing both of these effects simultaneously. Because of this integral feature, the model was able to provide a better estimation quality, which is justified by the lower values of the *AIC* information criterion.

7. Results & conclusion

In this study, we have proposed the GARCH-M-GJR-LEV model. Its integral feature is to simultaneously account for asymmetric responses in the returns and variance equations. According to Bollerslev (2022), investors perceive "good" and "bad" volatility periods differently, demanding different risk premiums. While well-known asymmetric GARCH models (e.g., EGARCH and GJR-GARCH) capture asymmetric responses of variance to shocks in returns, they either do not account for a risk premium or assume that it reacts equally to negative and positive return fluctuations. In contrast, while modeling the risk premium, the proposed method distinguishes between volatility caused by the "bear" and "bull" markets.

We derived the analytical expression of the unconditional variance for the proposed method and studied the properties via the simulated data analysis. According to the results, the GARCH-M-GJR-LEV model demonstrates a significant advantage compared with other methods in case an asymmetric relationship between the risk premium and volatility changes characterizes the data generating process. Therefore applying the model is necessary to obtain accurate estimates of GARCH model parameters, conditional volatilities, and returns.

Further, the introduced model was applied to study the volatility of the S&P 500 market index. As a result, we have found statistical evidence favoring the leverage effect in the mean equation over various periods. Consequently, investors demand a higher risk premium in case volatility is determined by negative shocks in returns rather than positive ones. Moreover, the results indicate that only negative shocks form a risk premium during some time intervals. In other words, investors demanded a risk premium only during "bad" volatility periods. This evidence coincides with different empirical results (Bollerslev et al., 2006; Rossi & Timmermann, 2015), and the prospect theory of Kahneman and Tversky (1979). Furthermore, we have found that a researcher may not be able to correctly estimate a risk premium without applying the GARCH-M-GJR-LEV model in case the data demonstrate a leverage effect in the equation of returns. This problem may lead to a significant misinterpretation of results. Finally, the Akaike information criterion indicates that the proposed model is the best among alternatives during most periods.

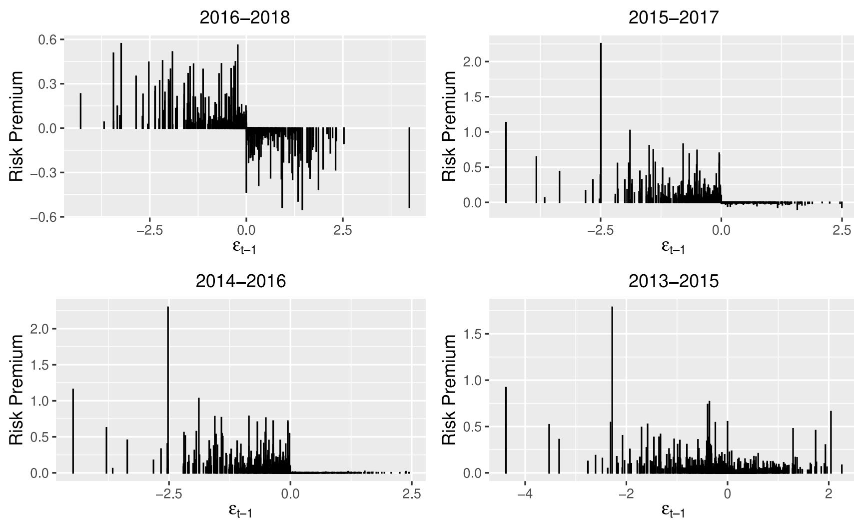

Fig. 3. Risk premium responses to shocks in returns.

CRediT authorship contribution statement

Juri Trifonov: Development of the main idea, Derivation of the unconditional variance, Drafting the manuscript, Software implementation of the proposed method. **Bogdan Potanin:** Drafting the manuscript, Software implementation of the proposed method.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Derivation of the unconditional variance

Here, we derive an analytical expression for the unconditional variance of returns, i.e., $Var\left(y_t\right)$.

Proof of Theorem 1. First, let us derive some preliminary results. Remind that from Glosten et al. (1993):

$$Var\left(\varepsilon_t\right) = \mathbb{E}\left[\varepsilon_t^2\right] = \mathbb{E}\left[\sigma_t^2\right] = \frac{\omega}{1-\alpha-\beta-\frac{1}{2}\gamma}. \tag{A.1}$$

Notice that $I_t = I_t^s$ for any $s \in N$ and σ_t is independent of ξ_t, so:

$$\mathbb{E}\left[I_t^2\sigma_t^2\right] = \mathbb{E}\left[I_t\sigma_t^2\right] = \underbrace{\mathbb{E}\left[\sigma_t^2|\xi_t<0\right]}_{\text{independent}} \times P\left(\xi_t<0\right) = \frac{1}{2}\mathbb{E}\left[\sigma_t^2\right], \tag{A.2}$$

$$\mathbb{E}\left[I_t^2\sigma_t^4\right] = \mathbb{E}\left[I_t\sigma_t^4\right] = \underbrace{\mathbb{E}\left[\sigma_t^4|\xi_t<0\right]}_{\text{independent}} \times P\left(\xi_t<0\right) = \frac{1}{2}\mathbb{E}\left[\sigma_t^4\right]. \tag{A.3}$$

Using the first and the fourth moments of standard normal distribution, we get:

$$\mathbb{E}\left[\varepsilon_t^2\right] = \mathbb{E}\left[\sigma_t^2\xi_t^2\right] = \mathbb{E}\left[\sigma_t^2\right] \times \mathbb{E}\left[\xi_t^2\right] = \mathbb{E}[\sigma_t^2], \tag{A.4}$$

$$\mathbb{E}\left[\varepsilon_t^4\right] = \mathbb{E}\left[\sigma_t^4\xi_t^4\right] = \mathbb{E}\left[\sigma_t^4\right] \times \mathbb{E}\left[\xi_t^4\right] = 3\mathbb{E}\left[\sigma_t^4\right]. \tag{A.5}$$

Simply expanding the expression of shocks, one gets:

$$\mathbb{E}\left[\varepsilon_t^2\sigma_t^2\right] = \mathbb{E}\left[\sigma_t^2\xi_t^2\sigma_t^2\right] = \mathbb{E}\left[\sigma_t^4\right]. \tag{A.6}$$

The distribution of ξ_t^2 does not depend on the event $\xi_t < 0$ if ξ_t is symmetric around zero, which is the case of a standard normal distribution, so:

$$\mathbb{E}\left[I_t\varepsilon_t^2\right] = \underbrace{\mathbb{E}\left[\sigma_t^2\xi_t^2|\xi_t<0\right]}_{\text{independent}} \times P\left(\xi<0\right) = \frac{1}{2}\mathbb{E}\left[\sigma_t^2\right] \times \mathbb{E}\left[\xi_t^2\right] = \frac{1}{2}\mathbb{E}\left[\sigma_t^2\right], \tag{A.7}$$

$$\mathbb{E}\left[I_t\varepsilon_t^4\right] = \underbrace{\mathbb{E}\left[\sigma_t^4\xi_t^4|\xi_t<0\right]}_{\text{independent}} \times P\left(\xi<0\right) = \frac{1}{2}\mathbb{E}\left[\sigma_t^4\right] \times \mathbb{E}\left[\xi_t^4\right] = \frac{3}{2}\mathbb{E}\left[\sigma_t^4\right], \tag{A.8}$$

$$\begin{aligned}\mathbb{E}\left[I_t\sigma_t^2\varepsilon_t^2\right] &= \mathbb{E}\left[I_t\sigma_t^4\xi_t^2\right] = \mathbb{E}\left[\sigma_t^4\xi_t^2|\xi_t<0\right] \times P\left(\xi_t<0\right) \\ &= \frac{1}{2}\mathbb{E}\left[\sigma_t^4\right] \times \mathbb{E}\left[\xi_t^2\right] = \frac{1}{2}\mathbb{E}\left[\sigma_t^4\right].\end{aligned} \tag{A.9}$$

Using the formulas above, it is straightforward to show that:

$$\begin{aligned}\mathbb{E}\left[\sigma_t^4\right] &= \mathbb{E}\left[\left(\omega+\alpha\varepsilon_{t-1}^2+\beta\sigma_{t-1}^2+\gamma I_{t-1}\varepsilon_{t-1}^2\right)^2\right] \\ &= \omega^2+\alpha^2\mathbb{E}\left[\varepsilon_{t-1}^4\right]+\beta^2\mathbb{E}\left[\sigma_{t-1}^4\right] \\ &\quad+\gamma^2\mathbb{E}\left[I_{t-1}^2\varepsilon_{t-1}^4+2\omega\alpha\mathbb{E}\left[\varepsilon_{t-1}^2\right]+2\omega\beta\mathbb{E}\left[\sigma_{t-1}^2\right]\right]+2\omega\gamma\mathbb{E}\left[I_{t-1}\varepsilon_{t-1}^2\right] \\ &\quad+2\alpha\beta\mathbb{E}\left[\varepsilon_{t-1}^2\sigma_{t-1}^2\right]+2\alpha\gamma\mathbb{E}\left[I_{t-1}\varepsilon_{t-1}^4\right]+2\beta\gamma\mathbb{E}\left[I_{t-1}\sigma_{t-1}^2\varepsilon_{t-1}^2\right] \\ &= \omega^2+\left(3\alpha^2+\beta^2+\frac{3}{2}\gamma^2+2\alpha\beta+3\alpha\gamma+\beta\gamma\right)\times\mathbb{E}\left[\sigma_t^4\right] \\ &\quad+\omega\mathbb{E}\left[\sigma_t^2\right]\times(2\alpha+2\beta+\gamma).\end{aligned} \tag{A.10}$$

Solving for $\mathbb{E}\left[\sigma_t^4\right]$, we get:

$$\mathbb{E}\left[\sigma_t^4\right] = \frac{\omega^2+\omega\mathbb{E}\left[\sigma_t^2\right]\times(2\alpha+2\beta+\gamma)}{1-3\alpha^2-\beta^2-\frac{3}{2}\gamma^2-2\alpha\beta-3\alpha\gamma-\beta\gamma}. \tag{A.11}$$

Now we are ready to derive the expression for the unconditional variance of y_t:

$$\begin{aligned}\sigma^2 = Var(y_t) &= Var\left(\mu+\lambda_1\sigma_{t-1}^2+\lambda_2 I_{t-1}\sigma_{t-1}^2+\varepsilon_t\right) \\ &= Var\left(\lambda_1\sigma_{t-1}^2+\lambda_2 I_{t-1}\sigma_{t-1}^2\right)+Var\left(\varepsilon_t\right) \\ &\quad+2\underbrace{Cov\left(\lambda_1\sigma_{t-1}^2+\lambda_2 I_{t-1}\sigma_{t-1}^2,\varepsilon_t\right)}_{\text{zero because of independence}} \\ &= \lambda_1^2 Var\left(\sigma_{t-1}^2\right)+\lambda_2^2 Var\left(I_{t-1}\sigma_{t-1}^2\right) \\ &\quad+2\lambda_1\lambda_2 Cov\left(\sigma_{t-1}^2, I_{t-1}\sigma_{t-1}^2\right)+\mathbb{E}\left[\sigma_t^2\right] \\ &= \lambda_1^2\times\left(\mathbb{E}\left[\sigma_{t-1}^4\right]-\mathbb{E}\left[\sigma_{t-1}^2\right]^2\right) \\ &\quad+\frac{1}{2}\lambda_2^2\times\left(\mathbb{E}\left[\sigma_{t-1}^4\right]-\frac{1}{2}\mathbb{E}\left[\sigma_{t-1}^2\right]^2\right) \\ &\quad+\lambda_1\lambda_2\times\left(\mathbb{E}\left[\sigma_{t-1}^4\right]-\mathbb{E}\left[\sigma_{t-1}^2\right]^2\right)+\mathbb{E}\left[\sigma_t^2\right].\end{aligned} \tag{A.12}$$

Note that since the process is stationary, then $\mathbb{E}\left[\sigma_{t-1}^4\right] = \mathbb{E}\left[\sigma_t^4\right]$ and $\mathbb{E}\left[\sigma_{t-1}^2\right] = \mathbb{E}\left[\sigma_t^2\right]$. Applying this and simplifying the expression, the final specification of the unconditional variance in the GARCH-M-GJR-LEV process takes the following form:

$$Var\left(y_t\right) = \left(\lambda_1^2 + \lambda_1\lambda_2\right) \times \left(\mathbb{E}\left[\sigma_t^4\right] - \mathbb{E}\left[\sigma_t^2\right]^2\right) + \frac{1}{2}\lambda_2^2 \times \left(\mathbb{E}\left[\sigma_t^4\right] - \frac{1}{2}\mathbb{E}\left[\sigma_t^2\right]^2\right) + \mathbb{E}\left[\sigma_t^2\right], \quad (A.13)$$

where $\mathbb{E}\left[\sigma_t^2\right]$ and $\mathbb{E}\left[\sigma_t^4\right]$ can be calculated by the formulas (A.1) and (A.11) correspondingly. □

Additionally, using the formulas mentioned above, it is possible to show that the unconditional variance of σ_t^2 is specified as follows:

$$Var(\sigma_t^2) = \left(\alpha^2 \times \left(3\mathbb{E}\left[\sigma_t^4\right] - \mathbb{E}\left[\sigma_t^2\right]^2\right) + 2\alpha\beta \times \left(\mathbb{E}\left[\sigma_t^4\right] - \mathbb{E}\left[\sigma_t^2\right]^2\right) + \gamma^2 \times \left(\frac{3}{2}\mathbb{E}[\sigma_t^4] - \frac{1}{4}\mathbb{E}\left[\sigma_t^2\right]^2\right) + \alpha\gamma \times \left(3\mathbb{E}[\sigma_t^4] - \mathbb{E}\left[\sigma_t^2\right]^2\right) + \beta\gamma \times \left(\mathbb{E}[\sigma_t^4] - \mathbb{E}\left[\sigma_t^2\right]^2\right)\right) \times \left(1 - \beta^2\right)^{-1}. \quad (A.14)$$

Appendix B. Alternative metrics for model comparison

In this Section, we provide the calculated values of *MAE* and *MSE* accuracy metrics for parameter estimates, conditional volatilities, and returns. Tables B.14–B.17, similarly to *RMSE*, demonstrate the advantage of the GARCH-M-GJR-LEV model over alternative methods.

Table B.14
Additional accuracy metrics of coefficient estimates (Set I).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$MAE(\hat{\mu})$	5.047	4.247	2.608
$MAE(\hat{\omega})$	2.501	2.281	2.092
$MAE(\hat{\alpha})$	9.239	3.014	2.908
$MAE(\hat{\beta})$	4.481	4.120	4.143
$MAE(\hat{\lambda}_1)$	19.959	5.473	5.340
$MAE(\hat{\gamma})$	–	21.676	5.341
$MAE(\hat{\lambda}_2)$	–	–	5.879
$MSE(\hat{\mu})$	0.500	0.364	0.211
$MSE(\hat{\omega})$	0.114	0.096	0.067
$MSE(\hat{\alpha})$	1.020	0.159	0.170
$MSE(\hat{\beta})$	0.374	0.343	0.289
$MSE(\hat{\lambda}_1)$	4.589	0.581	0.438
$MSE(\hat{\gamma})$	–	5.246	0.421
$MSE(\hat{\lambda}_2)$	–	–	0.790

Table B.15
Additional accuracy metrics of coefficient estimates (Set II).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$MAE(\hat{\mu})$	3.537	2.766	2.754
$MAE(\hat{\omega})$	1.725	1.357	1.377
$MAE(\hat{\alpha})$	12.245	2.528	2.580
$MAE(\hat{\beta})$	4.772	3.174	3.299
$MAE(\hat{\lambda}_1)$	13.063	26.252	25.080
$MAE(\hat{\gamma})$	–	14.681	23.254
$MAE(\hat{\lambda}_2)$	–	–	5.472
$MSE(\hat{\mu})$	0.176	0.108	0.114
$MSE(\hat{\omega})$	0.099	0.029	0.029
$MSE(\hat{\alpha})$	1.645	0.096	0.100
$MSE(\hat{\beta})$	0.505	0.168	0.157
$MSE(\hat{\lambda}_1)$	1.980	7.163	6.552
$MSE(\hat{\gamma})$	–	2.344	5.745
$MSE(\hat{\lambda}_2)$	–	–	0.490

Table B.16
Additional accuracy metrics of conditional volatilities and return predictions (Set I).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$MAE(\hat{\sigma})$	7.655	6.237	3.694
$MSE(\hat{\sigma})$	1.524	1.060	0.377
$MAE(\hat{y})$	71.677	71.453	69.117
$MSE(\hat{y})$	89.456	88.755	82.025

Table B.17
Additional accuracy metrics of conditional volatilities and return predictions (Set II).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$MAE(\hat{\sigma})$	9.161	4.478	3.957
$MSE(\hat{\sigma})$	1.766	0.481	0.337
$MAE(\hat{y})$	74.851	74.763	74.032
$MSE(\hat{y})$	100.787	100.572	97.548

Appendix C. Robustness to normality assumption violation

Here we provide the results for the QMLE estimators. Tables C.18–C.19 present the results assuming the Student's *t*-distribution, while Tables C.20–C.21 reflect the results considering the noncentral Student's *t*-distribution.

Table C.18
Accuracy metrics of coefficient estimates (Student's *t*-distribution).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$RMSE(\hat{\mu})$	8.961	7.596	5.076
$RMSE(\hat{\omega})$	5.962	3.881	3.275
$RMSE(\hat{\alpha})$	10.866	4.709	3.935
$RMSE(\hat{\beta})$	8.679	6.944	6.245
$RMSE(\hat{\lambda}_1)$	20.727	8.462	7.262
$RMSE(\hat{\gamma})$	–	22.989	7.666
$RMSE(\hat{\lambda}_2)$	–	–	11.212

Table C.19
Accuracy metrics and information criteria (Student's *t*-distribution).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$RMSE(\hat{\sigma})$	14.342	11.423	7.229
$Victories_{\sigma}\%$	2%	11%	87%
$RMSE(\hat{y})$	92.934	92.082	87.910
$Victories_{y}\%$	0%	1%	99%
AIC	2484.250	2468.050	2413.541
BIC	2508.789	2497.497	2447.896

Table C.20
Accuracy metrics of coefficient estimates (Noncentral Student's *t*-distribution).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$RMSE(\hat{\mu})$	9.309	8.144	5.804
$RMSE(\hat{\omega})$	7.633	7.206	6.827
$RMSE(\hat{\alpha})$	13.954	11.561	10.734
$RMSE(\hat{\beta})$	14.588	13.559	13.519
$RMSE(\hat{\lambda}_1)$	24.855	24.556	21.107
$RMSE(\hat{\gamma})$	–	28.285	9.677
$RMSE(\hat{\lambda}_2)$	–	–	15.128

Table C.21
Accuracy metrics and information criteria (Noncentral Student's *t*-distribution).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$RMSE(\hat{\sigma})$	13.147	12.676	10.230
$Victories_{\sigma}\%$	19%	16%	65%
$RMSE(\hat{y})$	83.456	83.335	80.786
$Victories_{y}\%$	5%	0%	95%
AIC	2318.284	2311.030	2270.023
BIC	2342.823	2340.477	2304.377

References

Awartani, B. M., & Corradi, V. (2005). Predicting the volatility of the S&P-500 stock index via GARCH models: the role of asymmetries. *International Journal of Forecasting*, *21*(1), 167–183. http://dx.doi.org/10.1016/j.ijforecast.2004.08.003.

Berkes, I., & Horváth, L. (2004). The efficiency of the estimators of the parameters in GARCH processes. *The Annals of Statistics*, *32*, 633–655. http://dx.doi.org/10.1214/009053604000000120.

Black, F. (1976). Studies of stock price volatility changes. In *Proceedings of the 1976 meeting of the business and economic statistics section* (pp. 177–181). American Statistical Association.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, *31*, 307–327. http://dx.doi.org/10.1016/0304-4076(86)90063-1.

Bollerslev, T. (2022). Realized semi(co)variation: Signs that all volatilities are not created equal. *Journal of Financial Econometrics*, *20*, 219–252. http://dx.doi.org/10.1093/jjfinec/nbab025.

Bollerslev, T., Litvinova, J., & Tauchen, G. (2006). Leverage and volatility feedback effects in high-frequency data. *Journal of Financial Econometrics*, *4*, 353–384. http://dx.doi.org/10.1093/jjfinec/nbj014.

Christie, A. (1982). The stochastic behavior of common stock variances value, leverage and interest rate effects. *Journal of Financial Economics*, *10*, 407–432. http://dx.doi.org/10.1016/0304-405X(82)90018-6.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, *50*, 987–1007. http://dx.doi.org/10.2307/1912773.

Engle, R. F., Lilien, D. M., & Robins, R. P. (1987). Estimating time varying risk premia in the term structure: The arch-m model. *Econometrica*, *55*, 391–407. http://dx.doi.org/10.2307/1913242.

Glosten, L. R., Jagannathan, R., & Runkle, D. E. (1993). On the relation between the expected value and the volatility of the nominal excess return on stocks. *The Journal of Finance*, *48*, 1779–1801. http://dx.doi.org/10.1111/j.1540-6261.1993.tb05128.x.

Hansen, P. R., Huang, Z., & Shek, H. H. (2012). Realized GARCH: a joint model for returns and realized measures of volatility. *Journal of Applied Econometrics*, *27*(6), 877–906. http://dx.doi.org/10.1002/jae.1234.

Hong, S. Y., & Linton, O. (2020). Nonparametric estimation of infinite order regression and its application to the risk-return tradeoff. *Journal of Econometrics*, *219*, 389–424. http://dx.doi.org/10.1016/j.jeconom.2020.03.009.

Kahneman, D., & Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, *47*, 263–292. http://dx.doi.org/10.2307/1914185.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*, *7*, 77–91. http://dx.doi.org/10.1111/j.1540-6261.1952.tb01525.x.

McAleer, M., & Hafner, C. M. (2014). A one line derivation of EGARCH. *Econometrics*, *2*(2), 92–97. http://dx.doi.org/10.3390/econometrics2020092.

McAleer, M., Hoti, S., & Chan, F. (2009). Structure and asymptotic theory for multivariate asymmetric conditional volatility. *Econometric Reviews*, *28*(5), 422–440. http://dx.doi.org/10.1080/07474930802467217.

Miralles-Marcelo, J. L., Miralles-Quirós, J. L., & del Mar Miralles-Quirós, M. (2013). Multivariate GARCH models and risk minimizing portfolios: The importance of medium and small firms. *The Spanish Review of Financial Economics*, *11*, 29–38. http://dx.doi.org/10.1016/j.srfe.2013.03.001.

Nelson, D. B. (1991). Conditional heteroskedasticity in asset returns: A new approach. *Econometrica*, *59*, 347–370. http://dx.doi.org/10.2307/2938260.

Rossi, A. G., & Timmermann, A. (2015). Modeling covariance risk in merton's ICAPM. *The Review of Financial Studies*, *28*, 1428–1461. http://dx.doi.org/10.1093/rfs/hhv015.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *The Journal of Finance*, *19*, 425–442. http://dx.doi.org/10.1111/j.1540-6261.1964.tb02865.x.

Zhang, X. F. (2006). Information uncertainty and stock returns. *The Journal of Finance*, *61*, 105–137. http://dx.doi.org/10.1111/j.1540-6261.2006.00831.x.